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B-CONVEX FUNCTIONS UNDER THE CONCEPT OF WITH AND WITHOUT DIFFERENTIABILITY

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Abstract

We introduce B-Convex functions under the concept of with and without differentiability. We establish theorems and examples on B-Convex functions with and without differentiability.

***Keywords*: B-Convex function with differentiable, B-Convex without differentiable.**

Introduction

Various generalization of convexity functions were appeared in the review. Sometimes, the driving force has been the fact that convexity plays a key role in optimization theory. Among many such contributions works of Bector, Mangasarian, recently Mond and Weir fall in this category. Many of the Works quoted above will support the claim that at times a class of functions built by weakening certain properties of convex functions. An interested reader may consult the most recent work of Bector that includes the works of many other researchers as well. B-Convex functions defined by relaxing the definition of a convex function. Similar functions were introduced by Bector as strong pseudo convex/concave function but no serious attempt made in utilizing the concepts like with and without differentiability in B-Convex functions. Hence in this chapter an attempt is made to fulfill the space in this aim of research by developing some theorems and methods to solve B-Convex function under the concept of with and without differentiability.

B-Convex functions definition under the concept of with-out differentiability

Assuming x be a not-null convexity subset of of R, R^n_+ Represent the set of non-negative real numbers

$$f : x \to R,\ g : x \to R\ b_1 X \times X \times [0, 1] \to R,\ b_2 : X \times X \times [0, 1] \to R$$

Suppose that x and y are in R^n so then

$$x \geq y \Leftrightarrow x_i \geq y_i \text{ for } 1 \leq i \leq n$$

$x \geq y \Leftrightarrow x_i \geq y_i$ for $1 \leq i \leq n$ and $x_s > y_s$ for some s

$1 \leq s \leq n$

$x > y \Leftrightarrow x_i > y_i$ for $1 \leq i \leq n$

B - Convex with respect to b1 and b2 if a point $x^o \in X$, the function f is stated as

$$f[\lambda x + (1-\lambda) x^o] \leq b_1 f(x) + b_2 f(x^o)$$

$\forall x \in X, \lambda \in [0, 1]$ with $b_1(x, x^o, \lambda) + b_2(x\, x^o, \lambda) = 1$,

$b_1(x, x^o, \lambda) \geq 0$

$b_2(x, x^o, \lambda) \geq 0$

Strictly B - Convex with respect to b_1 and b_2 if

$$f[\lambda x + (1-\lambda) x^0] < b_1 f(x) + b_2 f(x^0)$$

$$\forall x \in X, x \neq x^0, \lambda \in (0, 1), b_1 + b_2 = 1, b_1 b_2 \geq 0$$

B - Concave with respect to b_1 and b_2 if

$$f(\lambda x + (1-\lambda) x^o) \geq b_1 f(x) + b_2 f(x^o)$$

$$\forall x \in X, \lambda \in [0, 1], b_1 + b_2 = 1, b_1 b_2 \geq 0$$

Strictly B - Concave relative to b_1 and b_2 if

$$f(\lambda x + (1-\lambda) x) > b_1 f(x) + b f_2(x^o) \quad \forall x \in X, x \neq x^o,$$

$$\lambda \in (0, 1), b_1 + b_2 = 1, b_1 b_2 \geq 0$$

b - Linear with respect to b1 and b2 if it is both b - convex and b - concave with respect to b1and b 2.

Theorems on B - Convex functions under the concept of without differentiability:

We now prove the following theorems for the fraction of 2 appropriately restricted B- convex functions.

Suppose that f is convex and g is concave at $x^o \in X$. Further suppose that $f(x) \geq 0$, $g(x) > 0$ $\forall x \in X$ then $h = f/g$ is B - convex at $x^o \in X$ for the some b_1 and b_2

Proof: For any $0 \leq \lambda \leq 1$, for x and $x^o \in X$,

$$\frac{f}{g}[\lambda x + (1-\lambda) x^o] = \frac{f[\lambda x + (1-\lambda) x^o]}{g[\lambda x + (1-\lambda) x^o]}$$

Using the convexity and non-negativity and the concavity and positivity of g we have

$$\frac{f}{g}[\lambda x + (1-\lambda) x^o] \le \frac{\lambda f(x) + (1-\lambda) f(x^o)}{\lambda g(x) + (1-\lambda) g(x^o)}$$

$$= \left\{ [\lambda g(x)] \Big/ \lambda g(x) + (1-\lambda) g(x^o) \right\} \frac{f(x)}{g(x)} +$$

$$[(1-\lambda) g(x^o)] \Big/ [\lambda g(x) + (1-\lambda) g(x^o)] \Big\} \frac{f(x^o)}{g(x^o)}$$

$$\text{i.e., } \frac{f}{g}[\lambda x + (1-\lambda) x^o] \le b_1(x, x^o, \lambda) \frac{f(x)}{g(x)} +$$

$$b_2(x, x^o, \lambda) \frac{f(x^o)}{g(x^o)} \ \forall\, x \in \lambda,$$

$$\lambda \in [0, 1]$$

Where $b_1(x, x^0, \lambda) = \lambda g(x) / \lambda g(x) + (1-\lambda) g(x^0) \ge 0$

$b_2(x, x^o, \lambda) = [(1-\lambda) g(x^o) / [\lambda g(x) + (1-\lambda) g(x^o)$

$] \ge 0$ $b_1(x, x^o, \lambda) + b_2(x, x^o, \lambda) = 1$

This proves the result

The following example illustrates the above theorem.

$f(x) = 1 - 2x, 0 < x < 0.5$

$g(x) = 1 + 4x - 11x^2, 0 < x < 0.5$

Note that $f(x) \ge 0$, $g(x) > 0$ and f is convex and g is concave on $0 < x < 0.5$.

Now consider,

$$h(x) = \frac{f(x)}{g(x)} = \frac{1 - 2x}{1 + 4x - 11x^2} \quad 0 < x < 0.5$$

$$h^1(x) = \frac{(1 + 4x - 11x^2)(-2) - (1 - 2x)(4 - 22x)}{(1 + 4x - 11x^2)^2}$$

$$= \frac{-2 - 8x + 22x^2 - 4 + 22x + 8x - 44x^2}{(1 + 4x - 11x^2)^2}$$

$$\frac{-6 + 22x - 22x^2}{(1 + 4x - 11x^2)^2}$$

$$h^{11}(x) = \frac{[(1 + 4x - 11x^2)(1 + 4x - 11x^2)(22 - 44x) - (8 - 44x)(-6 + 22x - 22x^2)]}{(1 + 4x - 11x^2)^4}$$

$$= \frac{22 + 88x - 242x^2 - 44x - 176x^2 + 484x^3 + 48 - 176x + 176x^2 - 264x + 968x^2 - 968x^3}{(1 + 4x - 11x^2)^4}$$

$$= \frac{70 - 396x + 726x^2 - 484x^3}{(1 + 4x - 11x^2)^3}, \quad 0 < x < 0.5$$

$$h^{11}(x)(0.01) \simeq 58.96 > 0$$

$$h^{11}(0.4) \simeq -5.426 < 0$$

h is neither convex nor concave on o < x < 0.5 but it is B – Convex.

Some of the theorems on B - Convex functions under the concept of differentiability:

In this case we consider continuously differentiable function which are B-convex (strictly B - convex) with respect to b_1, b_2 where

$$b_2(x_1, x_2, \lambda) = 1 - b_1(x_1 x_2 \lambda), b_1 \geq 0$$

Further we assume that

$$b_1(x_1, x_2, \lambda) = \lambda\, b(x_1, x_2, \lambda)$$

Suppose that f is continuously differentiale and B - convex (with respect to b) at x^o x:

Then $(x - x_0)^t \nabla f(x^o \leq k(x, x^o)[f(x) - f(x^o)]$

$\forall x \in x$ and $k(x, x^o) = \underset{\lambda \to 0}{Lt}\ b(x, x^o, \lambda)$

Proof: Since the functions f is B - convexity at $x \in X$ for $\lambda \in (0, 1)$ we have

$$f(\lambda x + (1-\lambda) x^o) \leq \lambda [b(x_1, x_2, \lambda)] f(x) + (1-\lambda) b(x_1, x_2, \lambda) f(x^o)$$

$$= f(x^o) + \lambda b(x_1, x_2, \lambda) [f(x) - f(x^o)]$$

Applying the mean value hypothesis to the L.H.S.

$$f(x^o) + \lambda (x - x^o)^t \nabla f(x^o + \lambda\theta (x - x^o)) \leq f(x^o) + \lambda b(x, x^o, \lambda) [f(x) - f(x^o)]$$

$$0 \leq \theta \leq 1$$

$$\therefore (x - x^o)^t \nabla f(x^o + \lambda\theta (x - x^o) \leq b(x, x_o, \lambda)$$

$$[f(x) - f(x^o)], 0 \leq \lambda \leq 1$$

Taking limits on both sides as $\lambda \longrightarrow 0$,

We have,

$$(x - x_0)^t \nabla f(x^o \leq k(x, x^o) [f(x) - f(x^o)]$$

Where $k(x, x^o) = \underset{\lambda \to 0}{Lt} \; b(x, x^o, \lambda)$

Hence the proof

Theorem: Suppose that there exists a function kp : x x x R+ (set of +ve real's) such that for a continuously differentiable function f

$$K_p(x, x^o) [f(x) - f(x^o)] \geq (x - x^o)^t \nabla f(x^o)$$

$\forall$ x, x^o is an open convex set on x. So that f is B - convex function on x relative to b_1 and b_2 functions

Proof:

Let $x^o = \lambda x_1 + (1-\lambda) x_2$ where $x, x_2 \in x$ and

$$0 \leq \lambda \leq 1$$

$$K_p(x_1, \lambda x_1 + (1-\lambda) x_2) [f(x_1) - f(\lambda x_1 + (1-\lambda) x_2)] \geq (1-\lambda)(x_1 - x_2)^t \nabla$$
$$f(x_1) + (1-\lambda) x_2)] \quad ($$

$$\text{and } K_p(x_2, \lambda, x_1 + (1-\lambda) x_2) [f(x_2) - f(\lambda x_1 + (1-\lambda) x_2]$$
$$\geq -\lambda (x_1 - x_2)^t \nabla (\lambda x_1 + (1-\lambda) x_2)$$

we have

$$\lambda K_p (x_1, \lambda x_1 + (1 - \lambda) x_2) [f (x_1) - f (\lambda x_1 + (1 - \lambda) x_2)] + (1 - \lambda) K_p (x_2, \lambda x_1 +$$

$$(1 - \lambda) x_2) \ [f (x_2) - f (\lambda x_1 + (1 - \lambda) x_2] \lambda f (\lambda x_1 + (1 - \lambda) x_2)] \qquad (2.6)$$

Dividing both sides of (2.6) by

$$[\lambda K_p (x_1, \lambda, x_1 + (1 - \lambda) x_2) (1 - \lambda) K_p (x_2, \lambda, x_1 + (1 - \lambda) x_2)]$$

and taking

$$b_1 (x_1, x_2, \lambda) = [\lambda K_p (x_1, \lambda x_1 + (1 - \lambda) x_2] \Big/ \{\lambda K_p (x_1, \lambda x_1 + (1 - \lambda) x_2 + (1 - \lambda) K_p (x_2, \lambda x_1 + (1 - \lambda) x_2 \}$$

$$b_2 (x_1, x_2, \lambda) = [(1 - \lambda) K_p (x_2, \lambda x_1 + (1 - \lambda) x_2)] \Big/ [\lambda K_p (x_1, \lambda x_1 + (1 - \lambda) x_2 + (1 - \lambda) K_p (x_2, \lambda x_1 + (1 - \lambda) x_2]$$

This complete the proof of the theorem.

Theorem : Suppose that 'f' is continuous differentiable on x and ∃a + ve function

$K_p : X \times X \to R_p$ such that

$(x_1, x_2)^t [K_p (x_2, x_1) \nabla f (x_2) - K_p (x_1, x_2) \nabla f (x)] \geq 0$

holds $\forall x_1, x_2 \in x$ then f is B - convex with respect to some $\overset{*}{Kp}$,

in the sense at theorem (2.42) defined on X x X.

Proof : By the mean value theorem, we have

$f (x_2) - f (x_1) \geq (x_1 - x_2)^t \nabla f (x_1 + \overset{*}{\lambda} (x_2 - x_1)$

for some $\overset{*}{\lambda}$, $0 < \overset{*}{\lambda} < 1$

using the hypothesis, with x_2 replaced by

$x_1 + \overset{*}{\lambda} (x_2 - x_1) = \overset{*}{\lambda} x_2 + (1 - \overset{*}{\lambda}) x_1 \in x$

in (2.12), we have

$\overset{*}{\lambda} (x_1 - x_2)^t [K_p (x_1 + \overset{*}{\lambda} (x_2 - x_1), x_1)$

$\nabla f (x_1 + \overset{*}{\lambda} (x_2 - x_1) - K_p (x_1, x_1 + \overset{*}{\lambda} (x_2 - x_1) \nabla f (x_1) \geq 0$

from (2.12) and (2.13) [after cancelling $\overset{*}{\lambda}$ in 2.13 and using the fact that Kp is +ve], it follows that

$K_p (x_1 + \overset{*}{\lambda} (x_2 - x_1), x_1) [f (x_2) - f (x_1)] \geq K_p [x_1, x_1 + \overset{*}{\lambda} (x_2 - x_1)]$
$(x_1 - x_2)^t \nabla f (x_1)$

$\overset{*}{K}_p (x_2, x_1) = K_p (x_1 + \overset{*}{\lambda} (x_2 - x_1), x_1) \Big/ K_p [x_1, x_1 + \overset{*}{\lambda} (x_2 - x_1)]$

and involving (2.4.2) we have the conclusion

2.4.5 Theorem : Suppose that f and g are continuously differentiable b - convex and B - concave functions respectively with respect to the same $K(x, x^o) = \underset{\lambda \to 0}{Lt}\, b(x, x^o, \lambda)$. Furhter suppose that b is continuous in its third argument at $\lambda = 0$ and $f \geq 0$, $g > 0$. Then h = f/g is B - convex with respect to some $K^*(x, x^o)$

Proof : For any $x, x^o \in x$, using the hypothesis on f and g in conjunction with theorem (2.4.1)

$$(x - x^o)^t \nabla h(x^o) = \frac{[g(x^o)(x - x^o)^t \nabla f(x^o) - f(x^o)(x - x^o)^t \nabla g(x^o)}{(g[(x^o)]^2}$$

$$\leq k(x, x^o)\, g(x)\, g(x^o)\, [h(x) - h(x^o)] \Big/ [g(x^o)]^2$$

$$[\overset{*}{k}(x, x^o)[h(x) - h(x^o)]]$$

$$\text{where } \overset{*}{k}(x, x^o) = \frac{k(x, x^o)\, g(x)}{g(x^o)}$$

Hence the proof.

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